Self-avoiding walks and connective constants in small-world networks

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(Received 20 March 2003; published 11 August 2003)

Long-distance characteristics of small-world networks have been studied by means of self-avoiding walks (SAW's). We consider networks generated by rewiring links in one- and two-dimensional regular lattices. The number of SAW's u_n was obtained from numerical simulations as a function of the number of steps n on the considered networks. The so-called connective constant, $\mu=\lim_{n\to\infty}u_n/u_{n-1}$, which characterizes the long-distance behavior of the walks, increases continuously with disorder strength (or rewiring probability p). For small p, one has a linear relation $\mu=\mu_0+ap$, μ_0 and a being constants dependent on the underlying lattice. Close to $p=1$ one finds the behavior expected for random graphs. An analytical approach is given to account for the results derived from numerical simulations. Both methods yield results agreeing with each other for small p, and differ for p close to 1, because of the different connectivity distributions resulting in both cases.

DOI: 10.1103/PhysRevE.68.026106 PACS number(s): 02.50.−r, 05.40.Fb, 07.05.Mh, 05.50.+q

I. INTRODUCTION

Our world is formed by networks of different types (social, biological, technological, economic), whose characterization has launched in the past years the emergence of models incorporating the basic ingredients of real-life networks [1–3]. In particular, social networks form the substrate where processes such as information spreading or disease propagation take place. One expects that the structure of these complex networks will play an important role in such dynamical processes, which are usually studied by means of stochastic dynamics and random walks. Some processes, such as navigation and exploratory behavior are neither purely random nor totally deterministic, and can be also described by walks on graphs [4]. In this context, the generic properties of deterministic navigation [5] and directed self-avoiding walks [6] in random networks have been analyzed recently.

In the past years, networks displaying the ''small-world'' effect have been intensively studied [7–11]. Watts and Strogatz [7,12] proposed for this kind of networks a model based on a locally connected regular lattice, in which a fraction p of the links between nearest-neighbor sites are replaced by new random connections, thus creating long-range ''shortcuts.'' Hence, one has in the same network a local neighborhood (as for regular lattices) and some global properties of random graphs [13]. The small-world effect is usually measured by the scaling behavior of the characteristic path length ℓ, defined as the average of the distance between any two sites. In small-world networks, one has a logarithmic increase of ℓ with the network size, as happens for random graphs [2,13,14].

This short global length scale changes strongly the behavior of statistical physical problems on small-world networks, as compared with regular lattices (where one has $\ell\sim N^{1/d}$, N being the system size and d the lattice dimension). Among these problems, one finds signal propagation [7], spread of infections [15,16], and random spreading of information [17–19]. Site and bond percolation [16,20] as well as the Ising [21–23] and XY models [24] have been also studied in these networks. Most of the published work on small worlds has focused on networks obtained from one-dimensional lattices (rings). Small-world networks built by rewiring lattices of higher dimensions have been employed to study percolation, as a model of disease propagation [20,25]. Several characteristics of random walks on this kind of networks have been analyzed in connection with diffusion processes [26,27]. In particular, some properties of these walks, such as the probability of returning to the origin, were found to be intermediate between those corresponding to fractals and Cayley trees [28].

In this paper, we study self-avoiding walks in small-world networks built up from one- and two-dimensional regular lattices. A self-avoiding walk (SAW) is defined as a walk along the bonds of a given network which can never intersect itself. The walk is restricted to moving to a nearest-neighbor site during each step, and the self-avoiding condition constrains the walk to occupy only sites which have not been previously visited in the same walk [29]. SAW's have been used for modeling the large-scale properties of long-flexible macromolecules in solution [30], as well as for the study of polymers trapped in porous media, gel electrophoresis, and size exclusion chromatography, which deal with the transport of polymers through membranes with small pores [31]. They have also been employed to characterize complex crystal structures [32] and to analyze critical phenomena in lattice models [29,33]. Universal constants for SAW's have been discussed by Privman *et al.* [34].

The paper is organized as follows. In Sec. II, we give some basic definitions and concepts related to SAW's. In Sec. III, we present results for SAW's on simulated small-world networks, and in Sec. IV, we give an approximate analytical procedure to calculate the number of SAW's on this kind of networks. The paper closes with some conclusions in Sec. V.

II. BASIC DEFINITIONS

For regular lattices, the number u_n of different SAW's starting from a generic site has an asymptotic dependence for large n [34,35]:

$$u_n \sim n^{\gamma-1}\mu^n, \tag{1}$$

where γ is a critical exponent which depends on the lattice dimension, and μ is the so-called "connective constant" or effective coordination number of the considered lattice [35,36]. In general, for a lattice with coordination number (or connectivity) z, one has $\mu \leqslant z-1$. This parameter μ can be obtained from Eq. (1) by the limit

$$\mu = \lim_{n\to\infty} \frac{u_n}{u_{n-1}}. \tag{2}$$

The connective constant depends upon the particular topology of each lattice, and has been determined very accurately for two-dimensional (2D) and three-dimensional (3D) lattices [37,38]. In the following, we will consider Eq. (2) as a definition of the connective constant μ for any network. Note that the limit in Eq. (2) is well defined provided that the mean connectivity is finite. (Rigorous results on this question are given in Ref. [39].) Padé approximants and differential approximants [40,41] as well as Monte Carlo simulations [37] are alternative numerical methods to obtain asymptotic properties of self-avoiding walks.

For random and small-world networks, the number of SAW's of length n depends on the considered starting node of the network. In the sequel, we will call u_n the average number of SAW's of length n, i.e., the mean value obtained (for each n) by averaging over the network sites. For small-world networks, one expects μ values larger than that corresponding to the starting regular lattice. In particular, μ is expected to increase with p and approach the value corresponding to random lattices with mean connectivity z as $p \to 1$.

For regular lattices, all nodes have the same connectivity, i.e., the same number of nearest neighbors. However, for $p > 0$ different connectivities m are possible, giving rise to a probability distribution for which analytic expressions have been found [19,21]. For a (large) random network with mean connectivity z, the connectivity distribution $P_{rd}(m)$ follows a Poisson law [2,13]:

$$P_{rd}(m) = \frac{z^m e^{-z}}{m!}. \tag{3}$$

For random networks, the connective constant can be obtained in a straightforward manner, due to the absence of correlations between links. For $n=1$, one has obviously $u_1^{rd} = z$. Now, given a generic node and a link going out from it, the connectivity distribution $Q_{rd}(m)$ for the other end of the link in a random graph is given by

$$Q_{rd}(m) = \frac{m}{z} P_{rd}(m). \tag{4}$$

Then, the average number of two-step SAW's is given by

$$u_2^{rd} = z \sum_{m>1} (m-1) Q_{rd}(m), \tag{5}$$

and we find $u_2^{rd} = z^2$. Using the same procedure for $n > 2$, one has

$$u_n^{rd} = z^n, \tag{6}$$

and thus for (large) random networks one finds $\mu = z$. In connection with this, we note that for a Bethe lattice (or Cayley tree) with connectivity z, the number of SAW's is given by $u_n^{BL} = z(z-1)^{n-1}$, and one has $\mu = z-1$ (see, e.g., Ref. [42]).

III. NUMERICAL SIMULATIONS

The networks studied here have been generated from three different regular lattices: 1D ring with coordination number $z=4$ and 6, and 2D square lattice ($z=4$). To construct our small-world networks, we consider in turn each of the bonds in the starting lattice and substitute it with a given probability p by a new bond. This means that one end of the bond (chosen at random) is changed to a new node taken randomly from the entire network. We impose three conditions: (1) no two nodes can have more than one bond connecting them, (2) no node can be connected by a link to itself, and (3) isolated sites (with zero links) are not allowed. This method keeps constant the total number of links in the rewired networks (and consequently the average connectivity z). The total number of rewired links is $\frac{1}{2} zpN$ on average.

For networks generated in the present way, there is a crossover size $N^* \sim p^{-1}$ that separates the large- and small-world regimes [43,44], and the small-world behavior appears for any finite value of p ($0<p<1$) as soon as the network is large enough. Our networks included 1×10^5 sites in one dimension and 300×300 sites for the 2D system, so that we were in the small-world regime (system size $N>N^*$). In the sequel, we will call L the side length of the considered lattices, i.e., $L=N^{1/d}$. Periodic boundary conditions were assumed. We note that our networks differ from those discussed by Watts and Strogatz [7] in that these authors left untouched $z/2$ links per site. For our simulated small-world networks, we have obtained the average number u_n of SAW's up to $n=21$. Since u_n increases with p, this maximum n was reduced to $n=14$ close to $p=1$, in order to carry out averages over nodes of the generated networks. These numbers of steps in the SAW's are sufficient to obtain the connective constant μ with enough accuracy for our present purposes. In fact, the larger is p, the faster the ratio u_n/u_{n-1} converges with n.

First of all, we calculate the mean-squared end-to-end distance of the walks on our small-world networks:

$$R_n^2 = \langle (\mathbf{r}_n - \mathbf{r}_0)^2 \rangle, \tag{7}$$

where $\langle\ \rangle$ indicates an average over n-step SAW's with different starting sites and for different network realizations with given p. Here, $\mathbf{r}_n$ refers to the position of site n in the

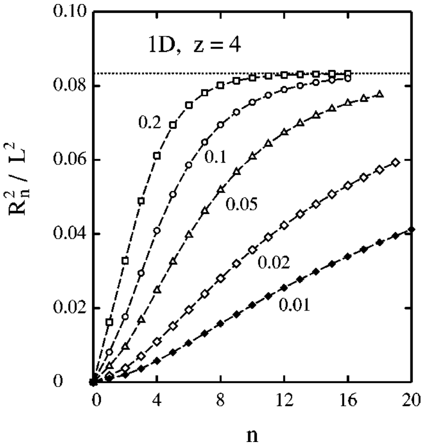

FIG. 1. Mean-squared end-to-end distance R_n^2 for SAW's on small-world networks generated from a 1D regular lattice with $N = 10^5$ sites and connectivity $z=4$. We present values for R_n^2, normalized by the squared system length L^2, as a function of the number of steps n. Different symbols correspond to several values of the rewiring probability p. From top to bottom: $p=0.2$, 0.1, 0.05, 0.02, and 0.01. A dotted line indicates the value $R_n^2/L^2=1/12$ corresponding to $p=1$. Dashed lines are guides to the eye.

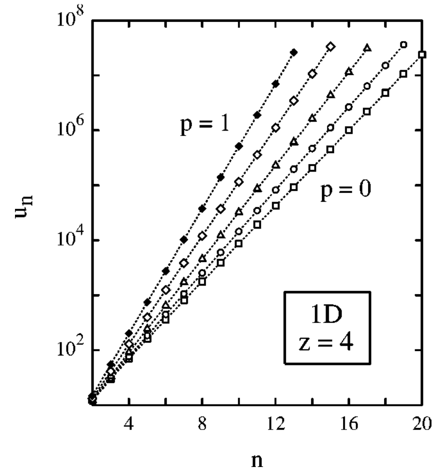

FIG. 2. Linear-log plot of the average number of self-avoiding walks u_n on small-world networks generated from 1D rings with connectivity $z=4$. We plot u_n as a function of the path length n for several rewiring probabilities p, as derived from numerical simulations. From top to bottom: $p=1$, 0.3, 0.1, 0.03, and 0. Dotted lines are guides to the eye.

d-dimensional Euclidean space of the underlying regular lattice, and $\mathbf{r}_0$ is the position of the origin for the considered walk. For SAW's on the 1D lattices considered here, one has $R_n^2 \sim bn^2$, whereas for the 2D square lattice $R_n^2 \sim bn^{3/2}$ [29,35,45], with a lattice-dependent constant b of the order of unity in all cases.

In Fig. 1, we present the ratio R_n^2/L^2 as a function of the number of steps n for SAW's on 1D small-world networks with $z=4$, for several values of the rewiring probability p. R_n^2 increases with n much faster than for the corresponding regular lattice due to the random connections introduced by the rewiring of links. (Note that for a regular lattice with $L = 10^5$, we have for $n=20$ a ratio $R_n^2/L^2 \sim 10^{-7}$.) In fact, if we call f_n the fraction of n-step paths that include at least one rewired link, we have

$$R_n^2 \approx b(1-f_n)n^2 + \tfrac{1}{12} f_n d L^2. \tag{8}$$

The first and second terms on the right-hand side come from SAW's without and with rewired links, respectively. The second term amounts, apart from fraction f_n, to the average Euclidean distance between any pair of sites in a d-dimensional box with side length L. Thus, in the limit of large networks (as those considered here, with $L \gg n$) one has $R_n^2 \approx f_n d L^2/12$. In the course of our numerical simulations, we have checked Eq. (8) by calculating independenlty f_n and R_n as a function of n. We found that both sides of this equation coincide within error bars (which are smaller than the symbol size in Fig. 1). Then, R_n^2 can be considered as a measure of the fraction of SAW's containing rewired links, and converges to $dL^2/12$ for large n in the rewired networks ($f_n \to 1$).

We now turn to the number u_n of SAW's on these networks. In Fig. 2, we show u_n as a function of the walk length for networks built up from a 1D lattice with coordination number $z=4$. We have plotted results for several rewiring probabilities p, from $p=0$ (regular lattice, squares) to $p=1$ (black diamonds). As expected, u_n increases as p is raised, since introducing long-range connections in the starting lattice opens new ways for the SAW's. In particular, such long-range links allow the walks to visit regions far away from the origin for small n, and thus avoid the constriction associated to move close to the starting site, which limits the number of possible self-avoiding walks. In the logarithmic plot of Fig. 2, one sees that $\log(u_n/u_{n-1})$ converges rather fast to a constant for each rewiring probability p, which allows us to calculate the corresponding connective constant. Also, from the results shown in Fig. 2 we find that for large n, the ratio between the number of SAW's for $p=1$ and $p=0$ increases as k^n with a constant $k=1.67$. This means that at long distances, for each link available for SAW's in the regular lattice, we have on average 1.67 connections for $p=1$. This number is, in fact, the ratio between connective constants for $p=1$ and $p=0$ in the case $d=1$, $z=4$.

The connective constant μ has been obtained for our simulated networks by finding the large-n limit of the ratio u_n/u_{n-1}. In Fig. 3, we present the resulting μ as a function of the rewiring probability p for our networks generated from 1D and 2D lattices. One observes that μ changes fast close to $p=0$, and the derivative $d\mu/dp$ decreases as p is raised. The largest change of μ in the whole region between $p=0$ and 1 is found for the networks with $z=6$. In fact, we find in this case an increase in μ of 2.13 to be compared with 1.48 and 1.07 in 1D and 2D rewired networks with $z=4$.

For $p=1$, our numerical procedure gives in all cases connective constants μ clearly lower than the mean connectivity z. In fact, we found $\mu=3.69$, 5.73, and 3.70 (± 0.01) for networks rewired from 1D lattices with z = 4 and 6, and from a 2D lattice ($z=4$), respectively. The obtained values for networks with $z=4$ coincide within error bars, irrespective of the starting 1D or 2D lattice, indicating that for p

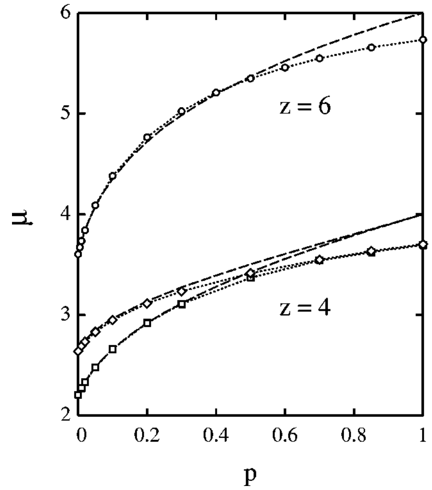

FIG. 3. Connective constant μ as a function of the rewiring probability p. Different symbols represent results obtained for small-world networks generated from 1D regular lattices with connectivities $z=4$ (squares) and $z=6$ (circles), as well as from a 2D square lattice (diamonds). Dotted lines are guides to the eye. Error bars are less than the symbol size. Results for μ obtained by means of the analytical method described in Sec. IV are plotted as dashed lines.

$=1$ the resulting rewired networks lost memory of the starting regular lattice. However, the μ values obtained for simulated networks with $p=1$ contrast with those expected for random networks, which coincide in each case with the average connectivity z, as explained above. This occurs because our networks with $p=1$ are not true (Poissonian) random networks, since they still keep memory of the starting regular lattices [19]. This memory effect is mainly due to the fact that one rewires only one end of each link, maintaining the other end on its original site. Hence, the connectivity distribution found for our rewired networks with $p=1$ does not coincide with that given above in Eq. (3). Such a difference with Poissonian random networks should be even stronger for small-world networks generated in a way similar to ours, but leaving untouched $z/2$ links per site, as those studied in earlier works [7,21].

To analyze the change of μ as a function of p for a given underlying lattice, we call $\Delta\mu=\mu-\mu_0$, μ_0 being the connective constant of the corresponding regular lattice. In Fig. 4, we show the obtained dependence of $\Delta\mu$ upon p for the considered 1D and 2D networks in a log-log plot. In all three cases, we find that $\Delta\mu$ can be fitted well by a power law $\Delta\mu\sim p^c$ for $p\lesssim 0.01$. For 1D networks, exponent c is found to be 0.98 ± 0.03 (for $z=4$) and 0.99 ± 0.03 (for $z=6$). For 2D networks, we found $c=0.99\pm0.03$. Thus, our results indicate a linear dependence $\mu=\mu_0+ap$ for small p, irrespective of the underlying lattice.

Therefore, the functional form for the p dependence of $\Delta\mu$ close to $p=0$ does not depend on the dimension of the starting regular lattice. This constrasts with other properties of small-world networks, which have been found to change as $p^{1/d}$, d being the dimension of the underlying lattice. This happens, for example, for the average number of nodes in "shell" n, which scales as $p^{1/d}$ [19]. Such a dependence is

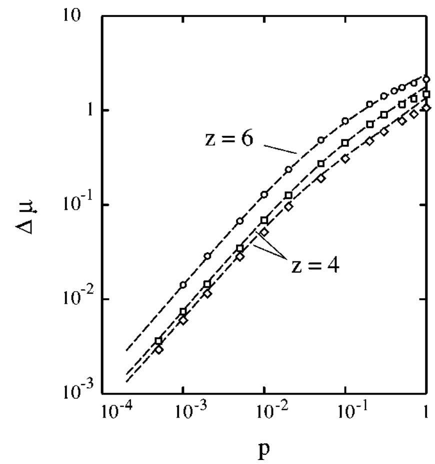

FIG. 4. Dependence upon the rewiring probability p of the change in connective constant, $\Delta\mu=\mu-\mu_0$, with respect to the corresponding regular lattices. Symbols represent the same networks as in Fig. 3. Dashed lines were obtained from analytical calculations by using Eq. (13).

related to the scaling of the characteristic length scale of small-world networks, namely, the average distance between the ends of shortcuts ξ, given by $\xi=(pz)^{-1/d}$ [20].

IV. ANALYTICAL APPROXIMATION

We now derive an approximate analytical expression that allows us to calculate u_n in a small-world network, assuming the sequence $\{u_n^0\}$ for the underlying regular lattice to be known [37,38]. For a given path length n, we obtain the mean number of SAW's u_n by considering all possible sequences of unrewired and rewired links in small-world networks. With this purpose, we calculate the probability of reaching a rewired link as a function of the walk length. In particular, we will obtain the conditional probability that the ith link in a SAW is a rewired one, assuming that link $i-1$ is an unrewired one $(i>1)$.

We first note that in a regular lattice, the ratio $c_i=u_i^0/u_{i-1}^0$ obviously depends on i. This ratio c_i measures the average number of available links starting from the $(i-1)$th site in a generic SAW on the underlying regular lattice, and allows us to calculate the number of possible unrewired links in a SAW on a rewired network. Taking into account that the fraction of unrewired connections in the whole rewired network is $1-p$, the mean number q_i of available unrewired links going out from site $i-1$ in a SAW is $q_i=c_i(1-p)$. On the other hand, the mean number of rewired links going out from an arbitrary node reached by an unrewired link is given by (see the Appendix)

$$w_i=p(z-\tfrac{1}{2}). \tag{9}$$

(Contrary to q_i, the mean number w_i is independent of i.) Therefore, the conditional probability that link $i(>1)$ is a rewired one, assuming that link $i-1$ is an unrewired one, is given by

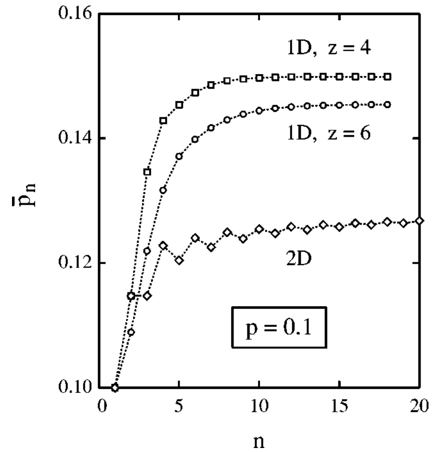

FIG. 5. Conditional probability $\overline{p_n}$ that the nth step in a SAW be a rewired link, assuming that link $n-1$ is an unrewired one, as derived for $p=0.1$ from the analytical procedure described in the text. $\overline{p_n}$ is presented as a function of step n for small-world networks built up from 1D rings with $z=4$ (squares) and $z=6$ (circles), as well as from a 2D square lattice (diamonds). Dotted lines are guides to the eye.

$$\overline{p_i} \equiv \frac{w_i}{w_i+q_i} = \frac{z'p}{z'p+c_i(1-p)}, \tag{10}$$

with $z'=z-1/2$. For $i=1$, we take $\overline{p_1}=p$. This probability $\overline{p_i}$ is shown in Fig. 5 for the three types of networks considered here, for a rewiring probability $p=0.1$.

Thus, the average number of i-step SAW's that do not include rewired connections is

$$A_i = (1-\overline{p_1}) \cdots (1-\overline{p_i}) u_i^0, \tag{11}$$

and the number of those consisting of $i-1$ unrewired links and a rewired one in step i is

$$B_i = (1-\overline{p_1}) \cdots (1-\overline{p_{i-1}}) \overline{p_i} u_i^0. \tag{12}$$

Now we note that a rewired connection in step i in a (large) small-world network ends on a random site of the network (most probably far away from the sites already visited in the same walk). This means that in step $i+1$, one begins most probably with a situation similar to that found in step $i=1$. Finally, the average number of n-step SAW's is given by

$$u_n = \sum_{i_1+\cdots+i_j=n} B_{i_1} B_{i_2} \cdots B_{i_{j-1}} A_{i_j}, \tag{13}$$

where the sum is extended to all possible combinations of indices $i_1, \ldots, i_j$ with sum equal to n, including null indices, for which we have $B_0=A_0=1$. Each term in Eq. (13) represents a sequence of unrewired and rewired links, and thus the sum includes 2^n terms. The SAW's corresponding to the general term in the sum include $j-1$ rewired links (in step number $i_1, i_1+i_2, i_1+\cdots+i_{j-1}$). As an example, Eq. (13) gives, for $n=2$, $u_2=B_2+B_1^2+B_1A_1+A_2$.

Note that the above equations, although rather accurate, are not exact. There are two reasons for this: First, each ratio c_i is an average number of allowed links starting from a site reached in $i-1$ steps, but the actual number of such allowed links depends on the particular site under consideration. Second, finite-size effects should appear, unless the considered networks are large enough.

Values of connective constants μ derived from u_n obtained with this procedure are presented in Figs. 3 and 4 as a function of the rewiring probability p (dashed lines). There appears to be good agreement with μ derived from numerical simulations (symbols) for $p<0.2$. For larger p, the connective constants deduced from the analytical method are larger than those yielded by the simulations. (For $z=6$ there is a region around $p=0.3$ where the analytical results are slightly lower than those found for the simulated networks.)

For $p=1$ our analytical procedure gives $\mu=z$, as for random networks. On the other hand, our numerical simulations for small-world networks gave in all cases in the limit $p=1$ connective constants μ clearly lower than the mean connectivity z, as indicated above and shown in Fig. 3. This difference between both procedures is due to the above-mentioned fact that simulated networks with $p=1$ do not have a Poissonian connectivity distribution, which is implicitly assumed in the analytical method for this value of p. As a matter of fact, in this case we have $A_i=0$ for $i>0$, $B_i=0$ for $i>1$, and $B_1=z$, giving in Eq. (13) $u_n=z^n$, as for random networks [see Eq. (6)]. In this sense, our analytical procedure to calculate the number of SAW's gives a better interpolation between regular lattices and random graphs.

In the context of this analytical approach, it is natural to expect close to $p=0$ a linear dependence of the connective constant μ on p, as found from our numerical simulations (see Sec. III), irrespective of the underlying lattice. By expanding the probability $\overline{p_i}$ [Eq. (10)] to first order in p, one finds for $i>1$: $\overline{p_i}=z'p/c_i+O(p^2)$. Introducing this expression for $\overline{p_i}$ into Eqs. (11)–(13), and keeping terms up to first order in p, we find

$$u_n = \sum_{i=0}^{n} B_i A_{n-i} \qquad (p \ll 1), \tag{14}$$

with the corresponding linearized expressions for B_i and A_{n-i}. This expression includes contributions of SAW's containing zero (term A_n, for $i=0$) and one rewired links (all other terms, $i=1,\ldots,n$). In this way, close to $p=0$ we find for the derivative $d\mu/dp$ the values 7.6 ($d=1, z=4$), 14.3 ($d=1, z=6$), and 6.3 ($d=2$). Thus, it is clear that the functional dependence of μ upon p for $p \ll 1$ does not change with the dimension of the underlying lattice.

V. CONCLUSIONS

Self-avoiding walks provide us with an adequate tool to study the long-range characteristics of small-world networks. For large networks, the number of SAW's increases asymptotically as $u_n \sim \mu^n$, provided that one considers system sizes $L \gg n$. For small-world networks generated from a given lattice, the effective connectivity μ ranges from the value of the regular lattice to $\mu = z$ for random graphs. For small p this effective connectivity follows a linear relation $\mu = \mu_0 + ap$, a being a constant dependent on the underlying lattice.

We have developed an analytical procedure to obtain the number of SAW's in small-world networks. This method is based on calculating probabilities of finding rewired or unrewired links in the walks, and gives results in good agreement with numerical simulations for $p \lesssim 0.2$. The results of both methods differ for larger p, since they assume in practice different connectivity distributions. Our analytical method gives in this respect a correct interpolation between regular lattices and random graphs. On the contrary, the rewiring (simulated) process gives rise to non-Poissonian connectivity distributions, even for a rewiring probability $p = 1$, yielding in this case networks with connective constants μ lower than the average connectivity z.

Both analytical calculations and simulations similar to those presented here can be useful to characterize other kinds of networks of current interest, such as scale-free networks, whose properties are known to depend on the asymptotic form of the connectivity distribution for large connectivities.

ACKNOWLEDGMENTS

The authors benefited from useful discussions with M. A. R. de Cara. This work was supported by CICYT (Spain) under Contract No. BFM2000-1318.

APPENDIX: DERIVATION OF CONDITIONAL PROBABILITIES

Here we calculate a conditional probability related to the number of possible connections starting from a generic node in a SAW, and necessary to derive Eq. (9) in our analytical approach in Sec. IV. In particular, we will obtain the average number of possible rewired links going out from a generic node X (so called for definiteness), assuming that this node was reached in a SAW through an unrewired link in step n. Here we mean by possible links all those connections that are available for step $n+1$ in a SAW (leading to nodes not previously visited in the same walk).

In a rewired network, the links with one end on node X can be classified for our present purpose into three types.

(1) Links which were not rewired and remain as in the original lattice. The probability distribution for the number r of these connections is given by

$$P_1(r) = \binom{z}{r} (1-p)^r p^{z-r}, \qquad r = 0, \ldots, z. \tag{A1}$$

(2) Rewired links for which the reference node X was not changed. Following the above notation, there are $z-r$ rewired links, from which s keeps one end on site X. The probability distribution for s is

$$P_2(s) = \binom{z-r}{s} \left(\frac{1}{2}\right)^{z-r}, \qquad s = 0, \ldots, z-r. \tag{A2}$$

(3) New (rewired) links arriving at site X. The distribution of the number v of these connections is (for large system size N)

$$P_3(v) = \frac{1}{v!} (tz)^v e^{-zt}, \qquad v \geqslant 0, \tag{A3}$$

with $t = p/2$.

Thus, given a site with r unrewired connections, the number of rewired links is $x = s + v$, where s depends on r and v is independent of r. The probability distribution for x is

$$Q_2^{(r)}(x) = \sum_{s=0}^{s_{max}} P_2(s) P_3(x-s), \tag{A4}$$

with $s_{max} = \min(z-r, x)$. Then, we have

$$Q_2^{(r)}(x) = \left(\frac{1}{2}\right)^{z-r} e^{-zt} \sum_{s=0}^{s_{max}} \binom{z-r}{s} \frac{1}{(x-s)!} (tz)^{x-s}. \tag{A5}$$

Hence, the probability distribution for the number of outgoing rewired links (assuming that the incoming link was an unrewired one) is

$$Q_{out}(x) = \sum_{r=1}^{z} Q_{in}(r) Q_2^{(r)}(x), \tag{A6}$$

where $Q_{in}(r)$ is the probability of reaching a node having r unrewired links:

$$Q_{in}(r) = \frac{r}{z} \binom{z}{r} (1-p)^{r-1} p^{z-r}. \tag{A7}$$

[$Q_{in}(r)$ is given, apart from a normalization constant, by the product $rP_1(r)$.]

Finally, the mean number of outgoing rewired links [calculated with the probability distribution $Q_{out}(x)$] is

$$\langle x \rangle = \sum_{x=1}^{\infty} x Q_{out}(x). \tag{A8}$$

Introducing expression (A6) into Eq. (A8), and after a straightforward but somewhat lengthy algebra, one finds the mean value

$$\langle x \rangle = p\left(z - \tfrac{1}{2}\right). \tag{A9}$$

This average value $\langle x \rangle$ is called w_i in the main text, and the last result is Eq. (9) there.

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